On Asymmetric Equilibria in Rent-Seeking Contests with Strictly Increasing Returns*

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Abstract. This paper revisits the n-player rent-seeking contest with homogeneous valuations and increasing returns. Our main result says that, for any $m \in \{2, \ldots, n-1\}$, there are threshold values $1 < R_*(m) < R^*(m) \leq 2$ for the Tullock parameter R such that a pure strategy equilibrium with m active players exists if and only if $R \in [R_*(m), R^*(m)]$. Among other things, this observation leads to a simple characterization of the values of R for which the n-player contest has a unique pure strategy equilibrium.

Keywords. Rent-seeking contests · Increasing returns · Asymmetric equilibria · Monotone comparative statics

JEL-Codes. C72 – Noncooperative Games; D72 – Political Processes: Rent-Seeking, Lobbying, Elections, Legislatures, and Voting Behavior; D74 – Conflict; Conflict Resolution; Alliances; Revolutions

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1 Introduction

In the n-player rent-seeking contest (Tullock, 1980), a given set of $n \geq 2$ players compete to get hold of a rent of value $V > 0$. The effort exerted by contestant $i \in \{1, \ldots, n\}$ is denoted by $x_i \geq 0$. Normalizing the value of the rent to unity, contestant i's payoff is given as

$$\Pi_i(x_1, \ldots, x_n) = \frac{x_i^R}{x_1^R + \ldots + x_n^R} - x_i,$$

where $R > 0$ is the usual parameter, and the ratio is read as $1/n$ if $x_1 = \ldots = x_n = 0$. In any *symmetric* equilibrium,[1]

$$x_1^* = \ldots = x_n^* = \frac{n-1}{n^2} R.$$

Moreover, this equilibrium exists if and only if $R \in (0, R^*(n)]$, where

$$R^*(n) = \frac{n}{n-1} \in (1, 2].$$

In an *asymmetric* equilibrium, however, a strict subset consisting of $m < n$ active players exert the same positive effort, while the remaining, inactive players exert zero effort (Pérez-Castrillo and Verdier, 1992). This type of equilibrium may arise if the contest technology exhibits increasing returns, i.e., if $R > 1$. Thus, the prediction here is a club of active rent-seekers, with outsiders being discouraged to even try to get hold of the rent. An asymmetric equilibrium is known to exist under two conditions, viz. that active players break even, and inactive players find it optimal to stay out. While the first condition is analogous to the parameter restriction for the symmetric equilibrium, the second condition is more complicated and represented by the rather "ugly" inequality

$$\frac{R^R \, (m-1)^R}{m^{2R-1}} \geq \frac{(R-1)^{R-1}}{R^R}. \tag{1}$$

[1] In this paper, we focus on equilibria in pure strategies.

Cornes and Hartley (2005) pointed out that, in the relevant domain, condition (1) becomes less stringent as m increases, which intuitively means that keeping outsiders out is easier for larger clubs. This observation leads to useful constraints on m under which an asymmetric equilibrium exists. However, it has to our knowledge not been formally studied how condition (1) depends on the contest technology. Thus, the set of parameter values R for which an asymmetric equilibrium with $m < n$ players exists has not really been well-understood so far.[2]

In this paper, we revisit the n-player Tullock contest with homogeneous valuations and strictly increasing returns. It is shown that, for any $m \in \{2, \ldots, n-1\}$, there exists a lower threshold value $R_*(m) \in (1, R^*(m))$ such that an asymmetric equilibrium with precisely $m < n$ active players exists if and only if $R \in [R_*(m), R^*(m)]$. Our contribution is, consequently, the formal proof that inequality (1) is monotone also with respect to R. Intuitively, with a larger R, competition for the rent within the club becomes tighter, which makes it even harder for outsiders to enter. Our main result therefore clarifies the nature of the conditions for the existence of asymmetric pure strategy equilibria in the n-player rent-seeking contest.

The analysis is extended in three ways. First, we show that the lower bound $R_*(m)$ is strictly declining in m. Given that the same is true for the upper bound $R^*(m)$, this means that the closed interval $[R_*(m), R^*(m)]$ in the parameter space over which asymmetric equilibria with precisely $m < n$ active players exist is shifting downwards as m goes up. Second, we show that

$$R^*(m+1) > R_*(m) \tag{2}$$

[2] For example, Ryvkin (2007, Sec. 3) offered valuable intuition and numerical illustration, though without formal proofs.

for all $m \geq 2$, which implies that the respective intervals in the parameter space over which an asymmetric equilibrium with precisely $m < n$ active players exists jointly cover the interval $[R_*(n-1), 2]$. Given that the symmetric equilibrium definitely exists for $R < R_*(n-1)$, these observations amount to an alternative proof for an important existence result for pure strategy equilibria in contests with increasing returns (Cornes and Hartley, 2005, Lem. 1). Third and finally, we derive the conditions on the parameter R for the existence of a unique pure strategy equilibrium in the n-player rent-seeking contest. In sum, these results provide a comprehensive characterization of the equilibrium set of the n-player rent-seeking contest with homogeneous valuations and increasing returns.

The remainder of this paper is structured as follows. Section 2 reviews prior work. Section 3 states our main result. Section 4 offers extensions. The Appendix contains a technical proof.

2 Review of Prior Work

Prior work characterized the best-response correspondence as well as the conditions for the existence of an asymmetric pure strategy equilibrium with $m < n$ active players.

Proposition 1 (**Pérez-Castrillo and Verdier, 1992; Cornes and Hartley, 2005**). *Suppose that $R > 1$. Then, the following holds:*

(i) Being active is a best response for contestant i if and only if

$$\sum_{j \neq i} x_j^R \in \left(0, \tfrac{(R-1)^{R-1}}{R^R}\right].$$

(ii) In any equilibrium with precisely m active players, $x_i^* = \frac{m-1}{m^2} R$, for any active contestant i.

(iii) An equilibrium with $m \in \{2, \dots, n-1\}$ active contestants exists if and only if $R \leq 2$ and

$$m \in \{m_*(R), \dots, m^*(R)\},$$

where $m_*(R)$ is the lowest integer satisfying inequality (1), and $m^*(R)$ is the largest integer satisfying $m \leq \frac{R}{R-1}$.

Proof. (i) See Pérez-Castrillo and Verdier (1992, Prop. 1). (ii) See Pérez-Castrillo and Verdier (1992, Prop. 3). (iii) See Cornes and Hartley (2005, Thm. 7). □

The proposition above allows to understand why relationship (1) captures the equilibrium condition for inactive players. The right-hand side of that inequality corresponds to the activity cutoff specified in part (i) of the proposition, while the left-hand side of that inequality corresponds to the aggregate $\sum_{i=1}^{m} x_i^R$ that results from the equilibrium efforts characterized in part (ii). Inactivity is optimal if the left-hand side weakly exceeds the right-hand side.

Proposition 1 is useful in these and other ways. However, as has been explained in the Introduction, the characterization of the equilibrium set accomplished by Proposition 1 remains partial because it does not allow to easily characterize the range of R for which an asymmetric equilibrium with m active players exists in the n-player rent-seeking contest.

3 Main Result

The main result of the present paper is the following.

Proposition 2. *There exists a lower threshold value $R_*(m) \in (1, R^*(m))$ such that an asymmetric equilibrium with precisely $m \in \{2, \dots, n-1\}$ active contestants exists if and only if $R \in [R_*(m), R^*(m)]$.*

Proof. As has been discussed in the Introduction, an asymmetric equilibrium with precisely $m \in \{2, \ldots, n-1\}$ active contestants exists in the n-player rent-seeking contest if and only if (i) active players break even, and (ii) inactive players find it optimal to stay out. The first condition amounts to $R \leq R^*(m)$. As for the second condition, suppose without loss of generality that contestants $i \in \{1, \ldots, m\}$ are active. Then, by Proposition 1,

$$x_1^* = \ldots = x_m^* = \frac{m-1}{m^2} R.$$

Therefore,

$$\sum_{i=1}^{m} x_i^R = \frac{(m-1)^R}{m^{2R-1}} R^R,$$

so that remaining inactive is optimal for any contestant $i \in \{m+1, \ldots, n\}$ if and only if inequality (1) holds true. Taking the logarithm on both sides, this inequality is seen to be equivalent to

$$2R \ln R + R \ln(m-1) - (R-1)\ln(R-1) - (2R-1)\ln m \geq 0.$$

In the limit $R \to 1$, we have $(R-1)\ln(R-1) \to 0$, so that the inequality fails to hold. On the other hand, at $R = R^*(m)$, any active contestant has a payoff of zero. Clearly, then, a passive contestant cannot profitably enter. Thus, the inequality holds strictly at $R = R^*(m)$. Note further that

$$\begin{aligned}
&\frac{\partial}{\partial R}\left(2R \ln R + R \ln(m-1) - (R-1)\ln(R-1) - (2R-1)\ln m\right) \\
&= 2\ln R - 2\ln m - \ln(R-1) + \ln(m-1) + 1 \\
&= \ln\left(\frac{R^2(m-1)e}{(R-1)m^2}\right),
\end{aligned}$$

where $e = \exp(1) \approx 2.71828$. To establish monotonicity, it therefore suffices to show that, in the relevant domain for R,

$$\frac{R^2(m-1)e}{(R-1)m^2} > 1.$$

We know that $m \leq \frac{R}{R-1}$, hence, it is sufficient to show that $\frac{R(m-1)}{m} > \frac{1}{e}$, which is clearly the case. Therefore, there indeed exists a threshold value $R_*(m) \in (1, R^*(m))$ with the stated property. $\square$

Table 1 shows the values of $R_*(m)$ and $R^*(m)$ for $m \in \{2, \ldots, 10\}$. For instance, an asymmetric equilibrium with $m = 3$ active players exists in a contest with $n > 3$ players if and only if $R \in [R_*(3), R^*(3)] = [1.16531, 1.5000]$.[3]

m	$R_*(m)$	$R^*(m)$
2	1.35050	2.00000
3	1.16531	1.50000
4	1.10848	1.33333
5	1.08079	1.25000
6	1.06438	1.20000
7	1.05352	1.16667
8	1.04580	1.14286
9	1.04002	1.12500
10	1.03555	1.11111

Table 1: Parameter ranges for asymmetric equilibria

4 Extensions

As extensions, we discuss the comparative statics (Subsection 4.1), an alternative proof of an existence result in Cornes and Hartley (2005) (Subsection 4.2), and conditions for the uniqueness of the equilibrium (Subsection 4.3).

[3]Notably, the symmetric equilibrium exists for any $R \leq R^*(n)$, i.e., even if $R < R_*(n)$. The reason for the relaxed conditions is that, in the case $n = m$, there are no potential entrants around which makes it easier to have the equilibrium.

4.1 Comparative Statics

As noted before, the upper bound $R^*(m) = \frac{m}{m-1}$ is strictly declining in m, starting from $R^*(2) = 2$ and approaching 1 as $m \to \infty$. The following result shows that the comparative statics of the lower bound is similar.

Proposition 3. *$R_*(m)$ is strictly declining in m, with $\lim_{m\to\infty} R_*(m) = 1$.*

Proof. To see why $R_*(m)$ is strictly monotone in m, note that $R = R_*(m)$ solves

$$2R \ln R + R \ln(m-1) - (R-1)\ln(R-1) - (2R-1)\ln m = 0.$$

Implicit differentiation shows that

$$\frac{dR_*(m)}{dm} = -\frac{1-(m-2)(R-1)}{m\,(m-1)\ln\frac{R^2(m-1)e}{(R-1)m^2}}.$$

Now, from $R = R_*(m) < R^*(m) = \frac{m}{m-1}$, it follows that $1-(m-2)(R-1) > 0$. By (3), also the denominator is positive. Therefore, $dR_*(m)/dm < 0$, as has been claimed. The limit property for $R_*(m)$ follows from $R_*(m) \in (1, R^*(m))$ and $\lim_{m\to\infty} R^*(m) = 1$. □

Essentially the same proof shows that $m_*(R)$ is strictly declining in R, which complements Proposition 1.

4.2 An Alternative Proof of Cornes and Hartley (2005, Lem. 1)

Cornes and Hartley (2005, Lem. 1) observed that, regardless of the number of players $n \geq 2$, a pure strategy equilibrium exists if and only if $R \leq 2$. The original proof is constructive. An alternative proof is presented below.

Proposition 4 (**Cornes and Hartley, 2005**). *An equilibrium exists in the n-player rent-seeking contest if and only if $R \leq 2$.*

Proof. Existence is standard for $R \leq 1$. Let, therefore, $R > 1$. Given Proposition 2, it suffices to show that inequality (2) holds, for any $m \geq 2$. This, however, is equivalent to checking that inequality (1) holds at $R = R^*(m+1) = \frac{m+1}{m}$ for any $m \geq 2$, which is a straightforward exercise detailed in the Appendix. □

4.3 Equilibrium Uniqueness

Cornes and Hartley (2005) derived the conditions under which the equilibrium in the n-player rent-seeking contest is unique. The following result restates those conditions more explicitly as a constraint on the parameter R.

Proposition 5. *In the n-player Tullock contest with homogeneous valuations, the symmetric equilibrium is the unique pure strategy equilibrium if and only if $R \in (0, R_*(n-1))$.*

Proof. The uniqueness of the pure strategy equilibrium for $R \leq 1$ is again standard. For $R > 1$, the symmetric equilibrium is unique if and only if, for any $m \in \{2, \ldots, n-1\}$, there is no asymmetric equilibrium, or using Proposition 2, if and only if $R \notin [R_*(m), R^*(m)]$. By Proposition 3, $R_*(n-1) \leq \ldots \leq R_*(2)$. Hence, for $R < R_*(n-1) < R^*(n)$, the symmetric equilibrium is indeed the unique equilibrium. Next, from, $R^*(2) = 2$ and the proof of Proposition 4, it follows that for any $R \in [R_*(n-1), 2]$, there exist asymmetric equilibria. This proves the claim. □

For illustration, consider again the simplest case where $n = 3$. The symmetric equilibrium exists for $R \leq R^*(3) = 1.50000$. By Proposition 2, however, an asymmetric equilibrium with two active players exists if and only if $R \in [R_*(2), R^*(2)] = [1.35050, 2.0000]$. Therefore, equilibrium uniqueness obtains if and only if $R < R_*(2) = 1.35050$.

A Appendix

This appendix contains details omitted from the proof of Proposition 4. We verify that the inequality

$$\frac{R^R\,(m-1)^R}{m^{2R-1}} \geq \frac{(R-1)^{R-1}}{R^R}$$

holds at $R = \frac{m+1}{m}$. Substituting gives

$$\frac{R^R(m-1)^R}{m^{2R-1}} = \frac{\left(\frac{m+1}{m}\right)^{\frac{m+1}{m}}(m-1)^{\frac{m+1}{m}}}{m^{\frac{m+2}{m}}} = \frac{(m^2-1)^{\frac{m+1}{m}}}{m^{\frac{2m+3}{m}}},$$

and

$$\frac{(R-1)^{R-1}}{R^R} = \frac{m^{-\frac{1}{m}}}{\left(\frac{m+1}{m}\right)^{\frac{m+1}{m}}} = \frac{m}{(m+1)^{\frac{m+1}{m}}}.$$

Hence, the inequality is equivalent to

$$\big((m^2-1)(m+1)\big)^{\frac{m+1}{m}} \geq m^{\frac{3m+3}{m}}.$$

Raising both sides to the power $\frac{m}{m+1}$ simplifies it to $(m^2-1)(m+1) \geq m^3$, which factors as $m(m-1) \geq 1$. Since this is true for $m \geq 2$, the inequality holds.

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